Characterizations for the fractional maximal operators on Carleson curves in local generalized Morrey spaces

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Abstract

In this paper we study the fractional maximal operator $\mathcal{M}^\alpha$ in the local generalized Morrey space $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ and the generalized Morrey space $M_{p,\varphi}(\Gamma)$ defined on Carleson curves Γ, respectively. For the operator $\mathcal{M}^\alpha$ we shall give a characterization the strong and weak Spanne-Guliyev type boundedness on $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ and the strong and weak Adams-Guliyev type boundedness on $M_{p,\varphi}(\Gamma)$.

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1 Introduction

Morrey spaces were introduced by C. B. Morrey [25] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory.

The main purpose of this paper is to establish the boundedness of fractional maximal operator $\mathcal{M}^\alpha$ in local generalized Morrey spaces $LM^{\{x_0\}}_{p,\varphi}(\Gamma)$ defined on Carleson curves Γ. We study Spanne-Guliyev type boundedness of the operator $\mathcal{M}^\alpha$ from $LM^{\{x_0\}}_{p,\varphi_1}(\Gamma)$ to $LM^{\{x_0\}}_{q,\varphi_2}(\Gamma)$, $1 < p < q < \infty$, and from the space $LM^{\{x_0\}}_{1,\varphi_1}(\Gamma)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2}(\Gamma)$, $1 < q < \infty$. Also we study Adams-Guliyev type boundedness of the operator $\mathcal{M}^\alpha$ from generalized Morrey spaces $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$, $1 < p < q < \infty$, and from the space $M_{1,\varphi}(\Gamma)$ to the weak space $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$, $1 < q < \infty$. We shall give a characterization for the Spanne-Guliyev and Adams-Guliyev type boundedness of the operator $\mathcal{M}^\alpha$ on the generalized Morrey spaces, including weak versions.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma =$ lengths of Γ. We denote

$$\Gamma(t,r) = \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

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where $B(t,r)=\{z\in\mathbb{C}:|z-t|<r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve if the condition

$$\nu\Gamma(t,r)\le c_0 r$$

holds for all $t\in\Gamma$ and $r>0$, where the constant $c_0>0$ does not depend on t and r. Let $L_p(\Gamma)$, $1\le p<\infty$ be the space of measurable functions on Γ with finite norm

$$\|f\|_{L_p(\Gamma)}=\left(\int_\Gamma |f(t)|^p d\nu(t)\right)^{1/p}.$$

Definition 2.1. Let $1\le p<\infty$, $0\le\lambda\le 1$, $[r]_1=\min\{1,r\}$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space, and by $\widetilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space, the set of locally integrable functions f on Γ with the finite norms

$$\|f\|_{L_{p,\lambda}(\Gamma)}=\sup_{t\in\Gamma,r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_p(\Gamma(t,r))},\qquad \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)}=\sup_{t\in\Gamma,r>0}[r]_1^{-\frac{\lambda}{p}}\|f\|_{L_p(\Gamma(t,r))}$$

respectively.

Note that (see [13, 15]) $L_{p,0}(\Gamma)=\widetilde{L}_{p,0}(\Gamma)=L_p(\Gamma)$,

$$\widetilde{L}_{p,\lambda}(\Gamma)=L_{p,\lambda}(\Gamma)\cap L_p(\Gamma)\quad\text{and}\quad \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)}=\max\{\|f\|_{L_{p,\lambda}(\Gamma)},\|f\|_{L_p(\Gamma)}\}$$

and if $\lambda<0$ or $\lambda>1$, then $L_{p,\lambda}(\Gamma)=\widetilde{L}_{p,\lambda}(\Gamma)=\Theta$, where Θ is the set of all functions equivalent to 0 on Γ.

We denote by $WL_{p,\lambda}(\Gamma)$ the weak Morrey space, and by $W\widetilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space, as the set of locally integrable functions f on Γ with finite norms

$$\|f\|_{WL_{p,\lambda}(\Gamma)}=\sup_{\beta>0}\beta\sup_{t\in\Gamma,r>0}\left(r^{-\lambda}\int_{\{\tau\in\Gamma(t,r):\,|f(\tau)|>\beta\}}d\nu(\tau)\right)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\Gamma)}=\sup_{\beta>0}\beta\sup_{t\in\Gamma,r>0}\left([r]_1^{-\lambda}\int_{\{\tau\in\Gamma(t,r):\,|f(\tau)|>\beta\}}d\nu(\tau)\right)^{1/p}.$$

Let $f\in L_1^{loc}(\Gamma)$. The fractional maximal operator $\mathcal{M}^\alpha$ and the potential operator $\mathcal{I}^\alpha$ on Γ are defined by

$$\mathcal{M}^\alpha f(t)=\sup_{t>0}(\nu\Gamma(t,r))^{-1+\alpha}\int_{\Gamma(t,r)}|f(\tau)|d\nu(\tau),$$

and

$$\mathcal{I}^\alpha f(t)=\int_\Gamma\frac{f(\tau)d\nu(\tau)}{|t-\tau|^{1-\alpha}},\quad 0<\alpha<1,$$

respectively.

Fractional maximal and potential operators in various spaces defined on Carleson curves has been widely studied by many authors (see, for example [3, 4, 19, 20, 21, 22, 23, 24, 26]).

N. Samko [26] studied the boundedness of the maximal operator $\mathcal{M} = \mathcal{M}^0$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces $L_{p,\lambda}(\Gamma)$:

Theorem A. *Let Γ be a Carleson curve, $1 < p < \infty$ and $0 \le \lambda \le 1$. Then $\mathcal{M}$ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.*

V. Kokilashvili and A. Meskhi [22] studied the boundedness of the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces and proved the following:

Theorem B. *Let Γ be a Carleson curve, $1 < p < q < \infty$, $0 < \alpha < 1$, $0 < \lambda_1 < \frac{p}{q}$, $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ are bounded from the spaces $L_{p,\lambda_1}(\Gamma)$ to $L_{q,\lambda_2}(\Gamma)$.*

The following Adams boundedness (see [1]) of the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ in Morrey space defined on Carleson curves was proved in [10].

Theorem C. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$ and $1 \le p < \frac{1-\lambda}{\alpha}$.*

1) If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$.

The following Adams boundedness of the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ in modified Morrey space $\widetilde{L}_{p,\lambda}(\Gamma)$ defined on Carleson curves was proved in [13], see also [15].

Theorem D. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$ and $1 \le p < \frac{1-\lambda}{\alpha}$.*

1) If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\alpha \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ from $\widetilde{L}_{p,\lambda}(\Gamma)$ to $\widetilde{L}_{q,\lambda}(\Gamma)$.

2) If $p = 1$, then the condition $\alpha \le 1 - \frac{1}{q} \le \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the operators $\mathcal{M}^\alpha$ and $\mathcal{I}^\alpha$ from $\widetilde{L}_{1,\lambda}(\Gamma)$ to $W\widetilde{L}_{q,\lambda}(\Gamma)$.

3 Local generalized Morrey spaces

We find it convenient to define the local generalized Morrey spaces in the form as follows, see [16].

Definition 3.1. Let $1 \le p < \infty$ and $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$. Fixed $t_0 \in \Gamma$, we denote by $LM_{p,\varphi}^{\{t_0\}}(\Gamma)$ $(WLM_{p,\varphi}^{\{x_0\}}(\Gamma))$ the local generalized Morrey space (the weak local generalized Morrey space), the space of all functions $f \in L_p^{\rm loc}(\Gamma)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}^{\{t_0\}}(\Gamma)} = \sup_{r>0} \frac{1}{\varphi(t_0,r)} \frac{1}{(\nu\Gamma(t_0,r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t_0,r))}$$

$$\Big(\|f\|_{WLM_{p,\varphi}^{\{t_0\}}(\Gamma)} = \sup_{r>0} \frac{1}{\varphi(t_0,r)} \frac{1}{(\nu\Gamma(t_0,r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t_0,r))}\Big).$$

Definition 3.2. Let $1 \le p < \infty$ and $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$. The generalized Morrey space $M_{p,\varphi}(\Gamma)$ is defined of all functions $f \in L_p^{loc}(\Gamma)$ by the finite norm

$$\|f\|_{M_{p,\varphi}} = \sup_{t\in\Gamma, r>0} \frac{1}{\varphi(t,r)} \frac{1}{(\nu\Gamma(t,r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t,r))}.$$

Also the weak generalized Morrey space $WM_{p,\varphi}(\Gamma)$ is defined of all functions $f \in L_p^{loc}(\Gamma)$ by the finite norm

$$\|f\|_{WM_{p,\varphi}} = \sup_{t\in\Gamma, r>0} \frac{1}{\varphi(t,r)} \frac{1}{(\nu\Gamma(t,r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t,r))}.$$

It is natural, first of all, to find conditions ensuring that the spaces $LM_{p,\varphi}^{\{t_0\}}(\Gamma)$ and $M_{p,\varphi}(\Gamma)$ are nontrivial, that is consist not only of functions equivalent to 0 on Γ.

Lemma 3.1. Let $t_0 \in \Gamma$ and $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$. If

$$\sup_{r<\tau<\infty} \frac{1}{\varphi(t_0,r)} \frac{1}{(\nu\Gamma(t_0,r))^{\frac{1}{p}}} = \infty \quad \text{for some } r>0, \tag{3.1}$$

then $LM_{p,\varphi}^{\{t_0\}}(\Gamma) = \Theta$.

Proof. Let (3.1) be satisfied and f be not equivalent to zero. Then $\|f\|_{L_p(\Gamma(t_0,r))} > 0$ for some $r > 0$, hence

$$\begin{aligned}\|f\|_{LM_{p,\varphi}^{\{t_0\}}} &\geq \sup_{r<\tau<\infty} \frac{1}{\varphi(t_0,\tau)} \frac{1}{(\nu\Gamma(t_0,\tau))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t_0,\tau))} \\ &\geq \|f\|_{L_p(\Gamma(t_0,r))} \sup_{r<\tau<\infty} \frac{1}{\varphi(t_0,\tau)} \frac{1}{(\nu\Gamma(t_0,\tau))^{\frac{1}{p}}}.\end{aligned}$$

Therefore $\|f\|_{LM_{p,\varphi}^{\{t_0\}}} = \infty$. Q.E.D.

Remark 3.3. We denote by $\Omega_{p,loc}$ the sets of all positive measurable functions φ on $\Gamma \times (0,\infty)$ such that for all $r > 0$,

$$\left\| \frac{1}{\varphi(t_0,\tau)} \frac{1}{(\nu\Gamma(t_0,\tau))^{\frac{1}{p}}} \right\|_{L_\infty(r,\infty)} < \infty.$$

In what follows, keeping in mind Lemma 3.1, for the non-triviality of the space $LM_{p,\varphi}^{\{t_0\}}(\Gamma)$ we always assume that $\varphi \in \Omega_{p,loc}$.

Lemma 3.2. Let $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$.

(i) If

$$\sup_{r<\tau<\infty} \frac{1}{\varphi(t,\tau)} \frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}} = \infty \quad \text{for some } r>0 \text{ and for all } t\in\Gamma, \tag{3.2}$$

then $M_{p,\varphi}(\Gamma) = \Theta$.

(ii) If

$$\sup_{0<\tau<r} \varphi(t,\tau)^{-1} = \infty \quad \text{for some } r>0 \text{ and for all } t\in\Gamma, \tag{3.3}$$

then $M_{p,\varphi}(\Gamma) = \Theta$.

Proof. (i) Let (3.2) be satisfied and f be not equivalent to zero. Then $\sup_{t\in\Gamma}\|f\|_{L_p(\Gamma(t,\tau))}>0$ for some $r>0$, hence

$$\|f\|_{M_{p,\varphi}} \geq \sup_{t\in\Gamma}\sup_{r<\tau<\infty}\frac{1}{\varphi(t,\tau)}\frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}}\|f\|_{L_p(\Gamma(t,\tau))}$$
$$\geq \sup_{t\in\Gamma}\|f\|_{L_p(\Gamma(t,\tau))}\sup_{r<\tau<\infty}\frac{1}{\varphi(t,\tau)}\frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}}.$$

Therefore $\|f\|_{M_{p,\varphi}}=\infty$.
(ii) Let $f\in M_{p,\varphi}(\Gamma)$ and (3.3) be satisfied. Then there are two possibilities:
Case 1: $\sup_{0<\tau<r}\varphi(t,\tau)^{-1}=\infty$ for all $r>0$.
Case 2: $\sup_{0<\tau<r}\varphi(t,\tau)^{-1}<\infty$ for some $s\in(0,r)$.

For Case 1, by Lebesgue differentiation theorem, for almost all $t\in\Gamma$,

$$\lim_{r\to 0+}\frac{\|f\chi_{\Gamma(t,r)}\|_{L_p}}{\|\chi_{\Gamma(t,r)}\|_{L_p}}=|f(t)|. \tag{3.4}$$

We claim that $f(t)=0$ for all those t. Indeed, fix t and assume $|f(t)|>0$. Then by (3.4) there exists $t_0>0$ such that

$$\frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}}\|f\|_{L_p(\Gamma(t,\tau))}\geq 2^{-1}c_2^{\frac{1}{p}}\,|f(t)|$$

for all $0<\tau\leq t_0$. Consequently,

$$\|f\|_{M_{p,\varphi}}\geq\sup_{0<\tau<t_0}\frac{1}{\varphi(t,\tau)}\frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}}\;\|f\|_{L_p(\Gamma(t,\tau))}\geq 2^{-1}c_2^{\frac{1}{p}}\,|f(t)|\sup_{0<\tau<t_0}\varphi(t,r)^{-1}.$$

Hence $\|f\|_{M_{p,\varphi}}=\infty$, so $f\notin M_{p,\varphi}(\Gamma)$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup_{s<\tau<\tau}\varphi(t,\tau)^{-1}=\infty$, hence

$$\sup_{s<\tau<\infty}\frac{1}{\varphi(t,\tau)}\frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}}\geq\sup_{s<\tau<r}\frac{1}{\varphi(t,\tau)}\frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}}$$
$$\geq\frac{1}{(\nu\Gamma(t,r))^{\frac{1}{p}}}\sup_{s<\tau<r}\varphi(t,\tau)^{-1}=\infty,$$

which is the case in (i). Q.E.D.

Remark 3.4. We denote by Ω_p the sets of all positive measurable functions φ on $\Gamma\times(0,\infty)$ such that for all $r>0$,

$$\sup_{t\in\Gamma}\left\|\frac{1}{\varphi(t,\tau)}\frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}}\right\|_{L_\infty(r,\infty)}<\infty,\quad\text{and}\quad\sup_{t\in\Gamma}\left\|\varphi(t,\tau)^{-1}\right\|_{L_\infty(0,r)}<\infty,$$

respectively. In what follows, keeping in mind Lemma 3.2, we always assume that $\varphi\in\Omega_p$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

Let $1 \le p < \infty$. Denote by $\mathcal{G}_p$ the the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto t^{\frac{1}{p}}\varphi(t) \in (0,\infty)$ is almost increasing.

Seemingly the requirement $\varphi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai established that there exists a function ρ such that ρ itself is decreasing, that $\rho(t)t^{n/p} \le \rho(T)T^{n/p}$ for all $0 < t \le T < \infty$ and that $LM_{p,\varphi}^{\{t_0\}}(\Gamma) = LM_{p,\rho}^{\{t_0\}}(\Gamma)$, $M_{p,\varphi}(\Gamma) = M_{p,\rho}(\Gamma)$.

By elementary calculations we have the following, which shows particularly that the spaces $LM_{p,\varphi}^{\{t_0\}}$, $WLM_{p,\varphi}^{\{t_0\}}$, $M_{p,\varphi}(\Gamma)$ and $WM_{p,\varphi}(\Gamma)$ are not trivial, see for example, [8, 9].

Lemma 3.3. Let $\varphi \in \mathcal{G}_p$, $1 \le p < \infty$, $\Gamma_0 = \Gamma(t_0, r_0)$ and χ_{Γ_0} is the characteristic function of the ball Γ_0, then $\chi_{\Gamma_0} \in LM_{p,\varphi}^{\{t_0\}}(\Gamma) \cap M_{p,\varphi}(\Gamma)$. Moreover, there exists $C > 0$ such that

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\Gamma_0}\|_{WLM_{p,\varphi}^{\{t_0\}}} \le \|\chi_{\Gamma_0}\|_{LM_{p,\varphi}^{\{t_0\}}} \le \frac{C}{\varphi(r_0)}$$

and

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\Gamma_0}\|_{WM_{p,\varphi}} \le \|\chi_{\Gamma_0}\|_{M_{p,\varphi}} \le \frac{C}{\varphi(r_0)}.$$

Proof. Let $\varphi \in \mathcal{G}_p$, $1 \le p < \infty$, $\Gamma_0 = \Gamma(t_0, r_0)$ denote an arbitrary ball in Γ. It is easy to see that

$$\|\chi_{\Gamma_0}\|_{WLM_{p,\varphi}^{\{t_0\}}} = \sup_{r>0} \frac{1}{\varphi(r)} \Big(\frac{|\Gamma(t_0,r) \cap \Gamma_0|}{\nu\Gamma(t_0,r)}\Big)^{1/p} \ge \frac{1}{\varphi(r_0)} \Big(\frac{|\Gamma_0 \cap \Gamma_0|}{|\Gamma_0|}\Big)^{1/p} = \frac{1}{\varphi(r_0)}.$$

Now, if $r \le r_0$, then $\varphi(r_0) \le C\varphi(r)$ and

$$\frac{1}{\varphi(r)} \Big(\frac{|\Gamma(t_0,r) \cap \Gamma_0|}{\nu\Gamma(t_0,r)}\Big)^{1/p} \le \frac{1}{\varphi(r)} \le \frac{C}{\varphi(r_0)}$$

for all $t \in \Gamma$.

On the other hand, if $r_0 \le r$, we have $\varphi(r_0)r_0^{1/p} \le C\varphi(r)r^{1/p}$ for all $t \in \Gamma$ and

$$\frac{1}{\varphi(r)} \Big(\frac{|\Gamma(t_0,r) \cap \Gamma_0|}{\nu\Gamma(t_0,r)}\Big)^{1/p} = \frac{|\Gamma(t_0,r) \cap \Gamma_0|^{1/p}}{c_2^{1/p}\varphi(r)r^{1/p}} \le \frac{|\Gamma_0|^{1/p}}{c_2^{1/p}\varphi(r)r^{1/p}} = \frac{r_0^{1/p}}{\varphi(r)r^{1/p}} \le \frac{C}{\varphi(r_0)}$$

for all $x \in \Gamma$. This completes the proof. Q.E.D.

4 Fractional maximal operator in the spaces $LM_{p,\varphi}^{\{t_0\}}(\Gamma)$ and $M_{p,\varphi}(\Gamma)$

In this section, we give a characterization for the Spanne-Guliyev type boundedness of the operator $\mathcal{M}^\alpha$ on the local generalized Morrey spaces $LM_{p,\varphi}^{\{t_0\}}(\Gamma)$ and the generalized Morrey spaces $M_{p,\varphi}(\Gamma)$, respectively, including weak versions. We give also a characterization for the Adams-Guliyev and Adams-Gunawan type boundedness of the operator $\mathcal{M}^\alpha$ on the generalized Morrey spaces $M_{p,\varphi}(\Gamma)$, including weak versions.

We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [5].

Theorem 4.1. Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$.

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{4.1}$$

4.1 Spanne-Guliyev type result

The following Guliyev local estimate for the fractional maximal operator $\mathcal{M}^\alpha$ is true, see for example, [2, 14].

Lemma 4.1. Let Γ be a Carleson curve, $1 \le p < q < \infty$, $0 \le \alpha < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha$ and $t_0 \in \Gamma$. Then for $p>1$ and any $r>0$ the inequality

$$\|\mathcal{M}^\alpha f\|_{L_p(\Gamma(t_0,r))} \lesssim \|f\|_{L_p(\Gamma(t_0,2r))} + r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-1} \|f\|_{L_1(\Gamma(t_0,\tau))} \tag{4.2}$$

holds for all $f \in L_p^{\rm loc}(\Gamma)$.

Moreover for $p=1$ the inequality

$$\|\mathcal{M}^\alpha f\|_{WL_1(\Gamma(t_0,r))} \lesssim \|f\|_{L_1(\Gamma(t_0,2r))} + r \sup_{\tau>2r} \tau^{-1} \|f\|_{L_1(\Gamma(t_0,\tau))} \tag{4.3}$$

holds for all $f \in L_1^{\rm loc}(\Gamma)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha$. For arbitrary ball $\Gamma(t_0, r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t_0,2r)}$ and $f_2 = f\chi_{\complement(\Gamma(t_0,2r))}$.

$$\|\mathcal{M}^\alpha f\|_{L_p(\Gamma(t_0,r))} \leq \|\mathcal{M}^\alpha f_1\|_{L_p(\Gamma(t_0,r))} + \|\mathcal{M}^\alpha f_2\|_{L_p(\Gamma(t_0,r))}.$$

By the continuity of the operator $\mathcal{M}^\alpha : L_p(\Gamma) \to L_q(\Gamma)$ from Theorem C we have

$$\|\mathcal{M}^\alpha f_1\|_{L_q(\Gamma(t_0,r))} \lesssim \|f\|_{L_p(\Gamma(t_0,2r))}.$$

Let y be an arbitrary point from $\Gamma(t_0, \tau)$. If $\Gamma(y, \tau) \cap \complement(\Gamma(t_0, 2r)) \neq \varnothing$, then $\tau > r$. Indeed, if $z \in \Gamma(y, \tau) \cap \complement(\Gamma(t_0, 2r))$, then $\tau > |y - z| \geq |t - z| - |t - y| > 2r - r = r$.

On the other hand, $\Gamma(y, \tau) \cap \complement(\Gamma(t_0, 2r)) \subset \Gamma(t_0, 2\tau)$. Indeed, $z \in \Gamma(y, \tau) \cap \complement(\Gamma(t_0, 2r))$, then we get $|t - z| \leq |y - z| + |t - y| < \tau + r < 2\tau$.

Hence

$$\begin{aligned}
\mathcal{M}^\alpha f_2(y) &= \sup_{\tau>0} \frac{1}{\big(\nu\Gamma(t_0, \tau)\big)^{1-\alpha}} \int_{\Gamma(y,\tau)\cap\complement(\Gamma(t_0,2r))} |f(z)| d\nu(z) \\
&\leq 2 \sup_{\tau>r} \frac{1}{\big(\nu\Gamma(t_0, 2\tau)\big)^{1-\alpha}} \int_{\Gamma(t_0,2\tau)} |f(z)| d\nu(z) \\
&= 2 \sup_{\tau>2r} \frac{1}{\big(\nu\Gamma(t_0, \tau)\big)^{1-\alpha}} \int_{\Gamma(t_0,\tau)} |f(z)| d\nu(z) \leq 2 \sup_{\tau>2r} \tau^{-1+\alpha} \int_{\Gamma(t_0,\tau)} |f(z)| d\nu(z).
\end{aligned}$$

Therefore, for all $y \in \Gamma(t_0, \tau)$ we have

$$\mathcal{M}^\alpha f_2(y) \leq 2 \sup_{\tau>2r} \tau^{-1+\alpha} \int_{\Gamma(t_0,\tau)} |f(z)| d\nu(z). \tag{4.4}$$

Thus

$$\|\mathcal{M}^\alpha f\|_{L_p(\Gamma(t_0,r))} \lesssim \|f\|_{L_p(\Gamma(t_0,2r))} + r^{\frac{1}{p}} \left(\sup_{\tau>2r} \tau^{-1+\alpha} \int_{\Gamma(t_0,\tau)} |f(z)| d\nu(z) \right).$$

Let $p = 1$. It is obvious that for any ball $\Gamma(t_0, r)$

$$\|\mathcal{M}^\alpha f\|_{WL_1(\Gamma(t_0,r))} \leq \|\mathcal{M}^\alpha f_1\|_{WL_1(\Gamma(t_0,r))} + \|\mathcal{M}^\alpha f_2\|_{WL_1(\Gamma(t_0,r))}.$$

By the continuity of the operator $\mathcal{M}^\alpha : L_1(\Gamma) \to WL_q(\Gamma)$ from Theorem C we have

$$\|\mathcal{M}^\alpha f_1\|_{WL_1(\Gamma)} \lesssim \|f\|_{L_1(\Gamma(t_0,2r))}.$$

Then by (4.4) we get the inequality (4.3).

Q.E.D.

Lemma 4.2. Let Γ be a Carleson curve, $1 \leq p < q < \infty$, $0 \leq \alpha < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha$ and $t_0 \in \Gamma$. Then for $p > 1$ and any $r > 0$ in Γ, the inequality

$$\|\mathcal{M}^\alpha f\|_{L_q(\Gamma(t_0,r))} \lesssim r^{\frac{1}{q}} \sup_{\tau>2r} \tau^{-\frac{1}{q}} \|f\|_{L_p(\Gamma(t_0,\tau))} \tag{4.5}$$

holds for all $f \in L_p^{\rm loc}(\Gamma)$.

Moreover for $p = 1$ the inequality

$$\|\mathcal{M}^\alpha f\|_{WL_1(\Gamma(t_0,r))} \lesssim r^{\frac{1}{q}} \sup_{\tau>2r} \tau^{-\frac{1}{q}} \|f\|_{L_1(\Gamma(t_0,\tau))} \tag{4.6}$$

holds for all $f \in L_1^{\rm loc}(\Gamma)$.

Proof. Let $1 < p < q < \infty$, $0 \le \alpha < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha$. Denote

$$\mathcal{M}_1 := r^{\frac{1}{q}} \sup_{\tau>2r} \tau^{-1+\alpha} \int_{\Gamma(t_0,r)} |f(z)| d\nu(z),$$
$$\mathcal{M}_2 := \|f\|_{L_p(\Gamma(t_0,2r))}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim r^{\frac{1}{q}} \sup_{\tau>2r} \tau^{-\frac{1}{q}} \left(\int_{\Gamma(t_0,\tau)} |f(z)|^p d\nu(z) \right)^{\frac{1}{p}}.$$

On the other hand,

$$r^{\frac{1}{q}} \sup_{\tau>2r} \tau^{-\frac{1}{q}} \left(\int_{\Gamma(t_0,\tau)} |f(z)|^p d\nu(z) \right)^{\frac{1}{p}} \gtrsim r^{\frac{1}{q}} \left(\sup_{\tau>2r} \tau^{-\frac{1}{q}} \right) \|f\|_{L_p(\Gamma(t_0,2r))} \approx \mathcal{M}_2.$$

Since by Lemma 4.1

$$\|\mathcal{M}^\alpha f\|_{L_p(\Gamma(t_0,r))} \le \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (4.5).

Let $p = 1$. The inequality (4.6) directly follows from (4.3). Q.E.D.

For the operator $\mathcal{M}^\alpha$ the following Spanne-Guliyev type result on the space $LM_{p,\varphi}^{\{t_0\}}(\Gamma)$ is valid (see [16]).

Theorem 4.2. Let Γ be a Carleson curve, $1 \le p < q < \infty$, $0 \le \alpha < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha$, $t_0 \in \Gamma$, $\varphi_1 \in \Omega_{p,loc}$, $\varphi_2 \in \Omega_{q,loc}$ and (φ_1, φ_2) satisfies the condition

$$\sup_{r<\tau<\infty} \tau^{\alpha-\frac{1}{p}} \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t_0, s)\, s^{\frac{1}{p}} \le C\, \varphi_2(t_0, r), \tag{4.7}$$

where C does not depend on r. Then for $p > 1$, the operator $\mathcal{M}^\alpha$ is bounded from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma)$ to $LM_{q,\varphi_2}^{\{t_0\}}(\Gamma)$ and for $p = 1$, the operator $\mathcal{M}^\alpha$ is bounded from $LM_{1,\varphi_1}^{\{t_0\}}(\Gamma)$ to $WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma)$.

Proof. By Theorem 4.1 and Lemma 4.2 we get

$$\begin{aligned}\|\mathcal{M}^\alpha f\|_{LM^{\{t_0\}}_{p,\varphi_2}(\Gamma)} &\lesssim \sup_{r>0} \varphi_2(t_0,r)^{-1} \sup_{\tau>r} \tau^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t_0,\tau))} \\ &\lesssim \sup_{r>0} \varphi_1(t,r)^{-1}\, r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t_0,r))} = \|f\|_{LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)},\end{aligned}$$

if $p \in (1,\infty)$ and

$$\begin{aligned}\|\mathcal{M}^\alpha f\|_{WLM^{\{t_0\}}_{p,\varphi_2}(\Gamma)} &\lesssim \sup_{r>0} \varphi_2(t_0,r)^{-1} \sup_{\tau>r} \tau^{-1} \|f\|_{L_1(\Gamma(t_0,r))} \\ &\lesssim \sup_{r>0} \varphi_1(t,r)^{-1}\, r^{-1} \|f\|_{L_1(\Gamma(t_0,r))} = \|f\|_{LM^{\{t_0\}}_{1,\varphi_1}(\Gamma)},\end{aligned}$$

if $p = 1$. Q.E.D.

From Theorem 4.2 we get (see [14]) the following

Corollary 4.1. Let Γ be a Carleson curve, $1 \le p < q < \infty$, $0 \le \alpha < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha$ and $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ satisfies the condition

$$\sup_{r<\tau<\infty} \tau^{-\frac{1}{q}} \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t,s)\, s^{\frac{1}{p}} \le C\, \varphi_2(t,r), \tag{4.8}$$

where C does not depend on t and r. Then for $p > 1$, the operator $\mathcal{M}^\alpha$ is bounded from $M_{p,\varphi_1}(\Gamma)$ to $M_{q,\varphi_2}(\Gamma)$ and for $p = 1$, the operator $\mathcal{M}^\alpha$ is bounded from $M_{1,\varphi_1}(\Gamma)$ to $WM_{q,\varphi_2}(\Gamma)$.

For proving our main results, we need the following estimate.

Lemma 4.3. Let Γ be a Carleson curve and $\Gamma_0 := \Gamma(t_0, r_0)$, then $r_0^\alpha \le C\mathcal{M}^\alpha \chi_{\Gamma_0}(t)$ for every $t \in \Gamma_0$.

Proof. It is well-known that

$$\mathrm{M}^\alpha f(t) \le 2^{1-\alpha} \mathcal{M}^\alpha f(t), \tag{4.9}$$

where $\mathrm{M}^\alpha(f)(t) = \sup_{B \ni t} |B|^{-1+\alpha} \int_B |f(\tau)| d\nu(\tau)$.

Now let $t \in \Gamma_0$. By using (4.9), we get

$$\begin{aligned}M_\alpha \chi_{\Gamma_0}(t) &\ge C\mathrm{M}_\alpha \chi_{\Gamma_0}(t) \ge C \sup_{B\ni t} |B|^{-1+\alpha} |B \cap \Gamma_0| \\ &\ge C|\Gamma_0|^{-1+\alpha} |\Gamma_0 \cap \Gamma_0| = C r_0^\alpha.\end{aligned}$$

Q.E.D.

The following theorem is one of our main results.

Theorem 4.3. Let Γ be a Carleson curve, $0 < \alpha < 1$, $t_0 \in \Gamma$ and $p, q \in [1, \infty)$.

1. If $1 \le p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, then the condition (4.7) is sufficient for the boundedness of the operator $\mathcal{M}^\alpha$ from $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ to $WLM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$. Moreover, if $1 < p < \frac{1}{\alpha}$, the condition (4.7) is sufficient for the boundedness of the operator $\mathcal{M}^\alpha$ from $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ to $LM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$.

2. If the function $\varphi_1 \in \mathcal{G}_p$, then the condition

$$r^{\alpha}\varphi_1(r) \le C\varphi_2(r), \tag{4.10}$$

for all $r > 0$, where $C > 0$ does not depend r, is necessary for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma)$ to $WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma)$ and $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma)$ to $LM_{q,\varphi_2}^{\{t_0\}}(\Gamma)$.

3. Let $1 \le p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. If $\varphi_1 \in \mathcal{G}_p$, then the condition (4.10) is necessary and sufficient for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma)$ to $WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma)$. Moreover, if $1 < p < \frac{Q}{\alpha}$, then the condition (4.10) is necessary and sufficient for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma)$ to $LM_{q,\varphi_2}^{\{t_0\}}(\Gamma)$.

Proof. The first part of the theorem proved in Theorem 4.2.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 4.3 we have $r_0^{\alpha} \le C\mathcal{M}^{\alpha}\chi_{\Gamma_0}(r)$. Therefore, by Lemma 3.3 and Lemma 4.3

$$r_0^{\alpha} \lesssim (\nu(\Gamma_0))^{-\frac{1}{p}} \|\mathcal{M}^{\alpha}\chi_{\Gamma_0}\|_{L_q(\Gamma_0)} \lesssim \varphi_2(r_0)\|\mathcal{M}^{\alpha}\chi_{\Gamma_0}\|_{M_{q,\varphi_2}} \lesssim \varphi_2(r_0)\|\chi_{\Gamma_0}\|_{M_{p,\varphi_1}} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}$$

or

$$r_0^{\alpha} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)} \text{ for all } r_0 > 0 \Longleftrightarrow r_0^{\alpha}\varphi_1(r_0) \lesssim \varphi_2(r_0) \text{ for all } r_0 > 0.$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. Q.E.D.

Remark 4.4. If we take $\varphi_1(r) = r^{\frac{\lambda-1}{p}}$ and $\varphi_2(r) = r^{\frac{\mu-1}{q}}$ at Theorem 4.3, then condition(4.10) is equivalent to $0 < \lambda < 1 - \alpha p$ and $\frac{\lambda}{p} = \frac{\mu}{q}$, respectively. Therefore, we get Theorem B from Theorem 4.3.

4.2 Adams-Guliyev type result

The following Guliyev pointwise estimate plays a key role where we prove our main results.

Theorem 4.5. Let Γ be a Carleson curve, $1 \le p < \infty$, $0 < \alpha < 1$ and $f \in L_p^{loc}(\Gamma)$. Then

$$\mathcal{M}^{\alpha} f(t) \le Cr^{\alpha}\, \mathcal{M}f(t) + C \sup_{r<s<\infty} s^{\alpha-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,s))} ds, \tag{4.11}$$

where C does not depend on f, $t \in \Gamma$ and $r > 0$.

Proof. Write $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t,2r)}$, $f_2 = f\chi_{\complement_{(\Gamma(t,2r))}}$. Then for all $z \in \Gamma$

$$\mathcal{M}^{\alpha} f(z) \le \mathcal{M}^{\alpha} f_1(z) + \mathcal{M}^{\alpha} f_2(z).$$

For $\mathcal{M}^{\alpha} f_1(t)$, following Hedberg's trick (see for instance [27], p. 354), for all $z \in \Gamma$ we obtain $|\mathcal{M}^{\alpha} f_1(z)| \le C_1 r^{\alpha}\mathcal{M}f(z)$. For $\mathcal{M}^{\alpha} f_2(z)$ with $z \in \Gamma(t, r)$ from (4.4) we have

$$\mathcal{M}^{\alpha} f_2(z) \le 2 \sup_{s>r} \tau^{-1+\alpha} \int_{\Gamma(t,s)} |f(z)|d\nu(z) \le C \sup_{r<s<\infty} s^{\alpha-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,s))}, \tag{4.12}$$

which proves (4.11). Q.E.D.

The following is a result of Adams-Guliyev type for the fractional integral on Carleson curves (see, [14]).

Theorem 4.6. (Adams-Guliyev type result) Let Γ be a Carleson curve, $1 \le p < q < \infty$, $0 < \alpha < \frac{1}{p}$ and let $\varphi \in \Omega_p$ satisfy condition

$$\sup_{r<\tau<\infty} \tau^{-1} \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi(t,s)\, s \le C\, \varphi(t,r), \tag{4.13}$$

and

$$\sup_{r<\tau<\infty} \tau^{\alpha} \varphi(t,\tau)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{4.14}$$

where C does not depend on $t \in \Gamma$ and $r > 0$. Then for $p > 1$, the operator $\mathcal{M}^{\alpha}$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ and for $p = 1$, the operator $\mathcal{I}^{\alpha}$ is bounded from $M_{1,\varphi}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi}(\Gamma)$. By Theorem 4.5 the inequality (4.11) is valid. Then from condition (4.14) and inequality (4.11) we get

$$\begin{aligned} \mathcal{M}^{\alpha} f(t) &\lesssim r^{\alpha}\, \mathcal{M}f(t) + \sup_{s>r} s^{\alpha-\frac{1}{p}} \, \|f\|_{L_p(\Gamma(t,s))} \\ &\le r^{\alpha}\, \mathcal{M}f(t) + \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \sup_{s>r} s^{\alpha}\, \varphi(t,s)^{\frac{1}{p}} \\ &\le r^{\alpha}\, \mathcal{M}f(t) + r^{-\frac{\alpha p}{q-p}} \, \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}. \end{aligned} \tag{4.15}$$

Hence choosing $r = \left(\frac{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}{\mathcal{M}f(t)}\right)$ for every $t \in \Gamma$, we have

$$\mathcal{M}^{\alpha} f(t) \lesssim (\mathcal{M}f(t))^{\frac{p}{q}} \, \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator $\mathcal{M}$ in $M_{p,\varphi}(\Gamma)$ provided by Theorem 4.2, in virtue of condition (4.13).

$$\begin{aligned} \|\mathcal{M}^{\alpha} f\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} &\lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \sup_{t\in\Gamma,\, r>0} \varphi(t,r)^{-\frac{p}{q}} r^{-\frac{1}{q}} \|\mathcal{M}f\|^{\frac{p}{q}}_{L_p(\Gamma(t,r))} \\ &\lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \|\mathcal{M}f\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned} \|\mathcal{M}^{\alpha} f\|_{WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)} &\lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \sup_{t\in\Gamma,\, r>0} \varphi(t,r)^{-\frac{1}{q}} r^{-\frac{1}{q}} \|\mathcal{M}f\|^{\frac{1}{q}}_{WL_1(\Gamma(t,r))} \\ &\lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \|\mathcal{M}f\|^{\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \lesssim \|f\|_{M_{1,\varphi}(\Gamma)}, \end{aligned}$$

if $p = 1 < q < \infty$. Q.E.D.

The following theorem is one of our main results.

Theorem 4.7. Let Γ be a Carleson curve, $0 < \alpha < 1$, $1 \le p < q < \infty$ and $\varphi \in \Omega_p$.

1. If $\varphi(t,r)$ satisfy condition (4.13), then the condition (4.14) is sufficient for the boundedness of the operator $\mathcal{M}^\alpha$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then the condition (4.14) is sufficient for the boundedness of the operator $\mathcal{M}^\alpha$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

2. If $\varphi \in \mathcal{G}_p$, then the condition

$$r^\alpha \varphi(r)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{4.16}$$

for all $r > 0$, where $C > 0$ does not depend r, is necessary for the boundedness of the operator $\mathcal{M}^\alpha$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ and from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

3. If $\varphi \in \mathcal{G}_p$, then the condition (4.16) is necessary and sufficient for the boundedness of the operator $\mathcal{M}^\alpha$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then the condition (4.16) is necessary and sufficient for the boundedness of the operator $\mathcal{M}^\alpha$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. The first part of the theorem is a corollary of Theorem 4.6.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 4.3 we have $r_0^\alpha \lesssim \mathcal{M}^\alpha \chi_{\Gamma_0}(t)$. Therefore, by Lemma 3.3 and Lemma 4.3 we have

$$\begin{aligned} r_0^\alpha &\lesssim (\nu(\Gamma_0))^{-\frac{1}{q}} \|\mathcal{M}^\alpha \chi_{\Gamma_0}\|_{L_q(\Gamma_0)} \lesssim \varphi(r_0)^{\frac{1}{q}} \|\mathcal{M}^\alpha \chi_{\Gamma_0}\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \\ &\lesssim \varphi(r_0)^{\frac{1}{q}} \|\chi_{\Gamma_0}\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \varphi(r_0)^{\frac{1}{q}-\frac{1}{p}} \end{aligned}$$

or

$$r_0^\alpha \varphi(r_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \text{ for all } r_0 > 0 \Longleftrightarrow r_0^\alpha \varphi(r_0)^{\frac{1}{p}} \lesssim r_0^{-\frac{\alpha p}{q-p}}.$$

Since this is true for every $t \in \Gamma$ and $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. Q.E.D.

4.3 Adams-Gunawan type result

The following is a result of Adams-Gunawan type for the fractional integral on Carleson curves (see, [17, 18]).

Theorem 4.8. (Adams-Gunawan type result). Let Γ be a Carleson curve, $0 < \alpha < 1$, $1 \le p < q < \infty$ and $\varphi \in \Omega_p$ satisfy condition (4.13) and

$$r^\alpha \varphi(t,r) + \int_r^\infty s^{\alpha-1} \varphi(t,s) ds \le C \varphi(t,r)^{\frac{p}{q}}, \tag{4.17}$$

where C does not depend on $t \in \Gamma$ and $r > 0$. Then for $p > 1$, the operator $\mathcal{M}^\alpha$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ and for $p = 1$, the operator $\mathcal{M}^\alpha$ is bounded from $M_{1,\varphi}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi}(\Gamma)$. By Theorem 4.5 the inequality (4.11) is valid. Then from condition (4.14) and inequality (4.11) we get

$$\begin{aligned} \mathcal{M}^\alpha f(t) &\lesssim r^\alpha \mathcal{M} f(t) + \sup_{s>r} s^{\alpha-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,s))} \\ &\le r^\alpha \mathcal{M} f(t) + \|f\|_{M_{p,\varphi}(\Gamma)} \sup_{s>r} s^\alpha \varphi(t,s). \end{aligned} \tag{4.18}$$

Thus, by (4.17) and (4.18) we obtain

$$\begin{aligned}\mathcal{M}^{\alpha} f(t) &\lesssim \min\left\{\varphi(t,r)^{\frac{p}{q}-1}\mathcal{M}f(t), \varphi(t,r)^{\frac{p}{q}}\|f\|_{M_{p,\varphi}(\Gamma)}\right\} \\ &\lesssim \sup_{r>0}\min\left\{r^{\frac{p}{q}-1}\mathcal{M}f(t), r^{\frac{p}{q}}\|f\|_{M_{p,\varphi}(\Gamma)}\right\} = (\mathcal{M}f(t))^{\frac{p}{q}}\,\|f\|^{1-\frac{p}{q}}_{M_{p,\varphi}(\Gamma)},\end{aligned} \tag{4.19}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 4.2 and (4.19), we get

$$\|\mathcal{M}^{\alpha} f\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}\,\|\mathcal{M}f\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)},$$

if $1 < p < q < \infty$ and

$$\|\mathcal{M}^{\alpha} f\|_{WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}(\Gamma)}\,\|\mathcal{M}f\|^{\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \lesssim \|f\|_{M_{1,\varphi}(\Gamma)},$$

if $p = 1 < q < \infty$. Q.E.D.

The following theorem is one of our main results.

Theorem 4.9. Let Γ be a Carleson curve, $0 < \alpha < 1$, $1 \le p < q < \infty$ and $\varphi \in \Omega_p$.

1. If $\varphi(t,r)$ satisfy condition (4.13), then the condition (4.17) is sufficient for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then the condition (4.17) is sufficient for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.
2. If $\varphi \in \mathcal{G}_p$, then the condition

$$r^{\alpha}\varphi(r)^{\frac{1}{p}} \le C\varphi(r)^{\frac{1}{q}}, \tag{4.20}$$

for all $r > 0$, where $C > 0$ does not depend r, is necessary for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ and from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

3. If $\varphi \in \mathcal{G}_p$, then the condition (4.20) is necessary and sufficient for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then the condition (4.20) is necessary and sufficient for the boundedness of the operator $\mathcal{M}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. The first part of the theorem is a corollary of Theorem 4.8.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 4.3 we have $r_0^{\alpha} \le C\mathcal{M}^{\alpha}\chi_{\Gamma_0}(t)$. Therefore, by Lemma 3.3 and Lemma 4.3 we have

$$\begin{aligned} r_0^{\alpha} &\lesssim (\nu(\Gamma_0))^{-\frac{1}{q}}\|\mathcal{M}^{\alpha}\chi_{\Gamma_0}\|_{L_q(\Gamma_0)} \lesssim \varphi(r_0)^{\frac{1}{q}}\|\mathcal{M}^{\alpha}\chi_{\Gamma_0}\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \\ &\lesssim \varphi(r_0)^{\frac{1}{q}}\|\chi_{\Gamma_0}\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \varphi(r_0)^{\frac{1}{q}-\frac{1}{p}}\end{aligned}$$

or

$$r_0^{\alpha}\varphi(r_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \text{ for all } r_0 > 0 \Longleftrightarrow r_0^{\alpha}\varphi(r_0)^{\frac{1}{p}} \lesssim \varphi(r_0)^{\frac{1}{q}}.$$

Since this is true for every $t \in \Gamma$ and $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. Q.E.D.

Remark 4.10. If we take $\varphi(r) = r^{\lambda-1}$ at Theorem 4.7, then the condition (4.16) is equivalent to $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$. Therefore, from Theorem 4.7 we get Theorem C.

Remark 4.11. If we take $\varphi(r) = [r]_1^{\lambda-1}$ at Theorem 4.7, then the condition (4.16) is equivalent to $\alpha \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{1-\lambda}$. Therefore, from Theorem 4.7 we get Theorem D.

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References

[1] D.R. Adams, *A note on Riesz potentials,* Duke Math. 42 (1975), 765-778.

[2] A. Akbulut, V.S. Guliyev and R. Mustafayev, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem., 137 (1) (2012), 27-43.

[3] A. Böttcher and Yu.I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators,* Basel, Boston, Berlin: Birkhäuser Verlag, 1997, 397 pages.

[4] A. Böttcher and Yu.I. Karlovich, *Toeplitz operators with PC symbols on general Carleson Jordan Curves with arbitrary Muckenhoupt Weights*, Trans. Amer. Math. Soc. 351 (1999), 3143-3196.

[5] V. Burenkov, A. Gogatishvili, V.S. Guliyev, R. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. 55 (8-10) (2010), 739-758.

[6] F. Deringoz, V.S. Guliyev, S.G. Samko, *Boundedness of maximal and singular operators on generalized Orlicz-Morrey spaces*, in Operator Theory, Operator Algebras and Applications, vol. 242 of Operator Theory : Advances and Applications, (2014), 1-24.

[7] Eridani, V. Kokilashvili, and A. Meskhi, *Morrey spaces and fractional integral operators*, Expo. Math. 27 (3) (2009), 227-239.

[8] Eridani, M. I. Utoyo, H. Gunawan, *A characterization for fractional integrals on generalized Morrey spaces*, Anal. Theory Appl. 28 (3) (2012), 263-268.

[9] A. Eroglu, V.S. Guliyev, C.V. Azizov, *Characterizations for the fractional integral operators in generalized Morrey spaces on Carnot groups*, Math. Notes 102 (5) (2017), 127-139.

[10] A. Eroglu, I.B. Dadashova, *Potential Operators on Carleson Curves in Morrey spaces*, Communications Faculty of Sciences University Of Ankara Series A1: Mathematics and Statistics, 66 (2) (2018), 188-194.

[11] A. Eroglu, J.V. Azizov, V.S. Guliyev, *Fractional maximal operator and its commutators in generalized Morrey spaces on Heisenberg group*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 44 (2) (2018), 304-317.

[12] A. Eroglu, G.A. Abasova, V.S. Guliyev, *Characterization of parabolic fractional integral and its commutators in parabolic generalized Orlicz-Morrey spaces*, Azerb. J. Math. 9 (1) (2019), 92-107.

[13] I.B. Dadashova, C. Aykol, Z. Cakir, A. Serbetci, *Potential operators in modified Morrey spaces defined on Carleson curves*, Trans. A. Razmadze Math. Inst. 172 (2018), no. 2, 1-15.

[14] V.S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, 1-20, 2009.

[15] V.S. Guliyev, J.J. Hasanov, Y. Zeren, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. 5 (4) (2011), 491-506.

[16] V.S. Guliyev, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) 193 (2) (2013), 211-227.

[17] V.S. Guliyev, Y. Sawano, *Linear and sublinear operators on generalized Morrey spaces with non-doubling measures*, Publ. Math. Debrecen 83 (3) (2013), 303-327.

[18] H. Gunawan, *A note on the generalized fractional integral operators*, J. Indones. Math. Soc. 9 (2003), 39-43.

[19] A.Yu. Karlovich, *Maximal operators on variable Lebesgue spaces with weights related to oscillations of Carleson curves*, Math. Nachr. 283 (1) (2010), 85-93.

[20] V. Kokilashvili, *Fractional integrals on curves*, (Russian) Trudy Tbliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 95 (1990), 56-70.

[21] V. Kokilashvili, A. Meskhi, *Fractional integrals on measure Spaces*, Frac. Calc. Appl. Anal. 4 (4) (2001), 1-24.

[22] V. Kokilashvili, and A. Meskhi, *Boundedness of Maximal and Singular Operators in Morrey Spaces with Variable Exponent*, Arm. J. Math. (Electronic) 1 (1) (2008), 18-28.

[23] V. Kokilashvili and S. Samko, *Boundedness of maximal operators and potential operators on Carleson curves in Lebesgue spaces with variable exponent*, Acta Math. Sin. (English Ser.) 24 (11) (2008), 1775-1800.

[24] V. Kokilashvili, S. Samko, *Sobolev theorem for potentials on Carleson curves in variable Lebesgue spaces*, Mem. Dierential Equations Math. Phys. 33 (2004), 157-158.

[25] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938), 126-166.

[26] N. Samko, *Weighted Hardy and singular operators in Morrey spaces*, J. Math. Anal. Appl. 350 (1) (2009), 56-72.

[27] E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.